Three-Dimensional Effects in Viscous Wakes

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Three-dimensionality in wakelike or jetlike free mixing may stem from initial geometric configurations, nonuniformities in flow variables over a cross section, or boundary conditions along the flow. These may be generated by bodies at angle of attack, nonaxisymmetric bodies, mixing of nonaxisymmetric jets with an outer flow, finite wings, or more artificial means. This paper is devoted to studies bearing on such configurations. The first section deals with the general mathematical model, in which the boundary layer approximations are used, and with methods of solution. Laminar and turbulent flow, compressibility, unsteadiness, and streamwise pressure gradients are admitted initially. The flux forms of the equations are given. Algebraic integrals of the energy equations and the diffusion (frozen-flow) equations are obtained. A simplification of the convective terms, roughly corresponding to the Oseen approximation, is used in the asymptotic downstream region. The second section contains explicit solutions for specific configurations, in particular for flows whose initial isovels are of elliptic shape. These flows may be wakelike or jetlike. Compressibility is admitted; however, the flows must have uniform pressure and must be steady. The final section deals with interpretation and evaluation of the results.

Nomenclature

D_l = associated with dissociation and ionization energies
e = eccentricity
E^2 = $[\epsilon^2 + 4s][1 + 4s]^{-1}$
h = static enthalpy ($h = \Sigma\alpha_i h_i$)
H = stagnation enthalpy, $H = h + u^2/2$
k^2 = $[1 + 4s]^{-1}$
K = pure constant of order 10^{-2}
L = constant characteristic length [see Eq. (26)]
Le = Lewis number
m = non-negative integer, $m = 0,1,2 \ldots$
Pr = Prandtl number
M_i = gram-mole weight of species i
R_0 = universal gas constant
Q = see Eq. (14a)
s,s_1 = transformed axial coordinate
t = time
T = absolute temperature
u = axial velocity component
$\tilde{u}$ = axial velocity defect ($\tilde{u} = u_e - u$)
v,w = crossflow velocity components (see Fig. 1)
W_i = net rate of production of species i
x = axial coordinate
y,z = cross-plane coordinate components (see Fig. 1)
α_i = mass friction of species i
δ = viscous layer thickness
δ_1 = viscous layer thickness in the plane $z = 0$
δ_2 = viscous layer thickness in the plane $y = 0$
$\bar{\delta}_m$ = minimum transform viscous layer thickness
ϵ = measure of the initial eccentricity [see Eq. (23)]
ϵ_v = turbulent eddy viscosity
η,η_1 = transformed normal coordinate
η_δ = value of η when $y = \delta$
Λ_j = vibrational energy of species j
μ = coefficient of (laminar) viscosity
ν = kinematic (laminar) viscosity ($\nu = \mu/\rho$)
ρ = density
ρ_r = reference density (may be a function of x)
σ,σ_1 = transformed normal coordinate
σ_δ = value of σ when $z = \delta$
φ_0 = $(u_e - u_0)/(u_e - u_{0c})$[see Eq. (25)]

Subscripts

c = conditions at an initial station
e,δ = local value at outer edge of viscous layer; can be equivalent to ∞
$t,x,y,z;$ $s,s_1,n,n_1,\sigma,\sigma_1$ = differentiation with respect to indicated variable
0 = conditions along the x axis
∞ = freestream undisturbed flow conditions

Superscripts

j = $j = 0$ for laminar flow; $j = 1$ for turbulent flow
m = non-negative integer, $m = 0, 1,2,\ldots$
bar = nondimensionalization with respect to external conditions $\bar{\rho} = \rho/\rho e$, $\mu = \mu/\mu e$, $\bar{\mu} = u/u_e$, etc.

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I. Introduction

THE aim of this paper is to continue an investigation started by Steiger and Bloom[1–3] concerning some characteristics of three-dimensional (nonaxisymmetric) wakes. Considerations are limited to flows in which the Prandtl boundary layer approximations may be made. That is, the crossflow velocities v,w are assumed small compared to the axial velocity u, and the crosswise pressure variations are assumed to have a negligible influence on the axial momentum balance. The governing equations are set down by familiar order of magnitude reasoning of boundary layer type.

The paper is divided into three main sections. The first includes the governing equations, algebraic integrals of the energy and diffusion equation as functions of the velocity (for Lewis and Prandtl numbers of unity and negligible chemical rates), a set of differential integral flux equations that involve only the axial velocity and the state of the gas, and solutions for both incompressible and compressible, laminar or turbulent flows including streamwise pressure gradients. The steady, incompressible constant pressure case, treated in Ref. 1, is extended here to admit pressure gradients. A crossflow solution also is examined briefly.

One type of mathematical simplicity in treating the elliptic, nonlinear equations is achieved by considering flows wherein decrements or increments with respect to the inviscid velocity u_e are sufficiently small to permit the linearization of the acceleration terms (excluding density). The linearization of these terms is in the sense of Oseen and others (see, e.g., Refs. 4 and 5). Second approximations, correctives for the effect of neglecting nonlinear acceleration terms, are discussed in Ref. 1.

It is important to observe that in the incompressible case the Oseen approximation completely linearizes the governing equations, and transform methods yield exact solutions for general types of initial conditions.

On the other hand, in compressible flow the governing equations remain nonlinear due to the density variations, and more approximate techniques must be used to achieve solutions of even the simplified equations. It can be shown that the useful density transformation, which is employed to simplify the solution of two-coordinate boundary layer problems, does not lead to simplification if applied rigorously to three-dimensional problems of the present type. However, it is suggested here that compressibility effects may be accounted for approximately by means of an analogous type of transformation. This transformation is approximate in the sense that it reduces to incompressible form the momentum integral flux condition and the nonlinear axial momentum equation evaluated along the x axis, rather than the differential equations at all points in the flow. The Oseen approximation permits the initial profiles to be wakelike, with velocities less than those of the freestream, or jetlike, with velocities in excess of those in the freestream; however, it excludes the case of a pure jet in a motionless ambient.

The second part of the paper presents solutions for special configurations, namely, elliptic wakelike or jetlike free mixing for isobaric, steady, compressible flows. Finally, the third part deals with the interpretation and evaluation of the solution of the elliptic wake. Additional work concerning three-dimensional wake effects of swirl and yaw is described in Refs. 6 and 7.

II. General Consideration of the Fluid Mechanics

A. Governing Equations

A rectangular Cartesian coordinate system (x,y,z) is adopted, as depicted in Fig. 1, with the x,y,z components of the velocity being u,v, and w, respectively.

The boundary layer equations assumed to govern are

Continuity

$$\rho_t + (\rho u)_x + (\rho v)_y + (\rho w)_z = 0 \tag{1}$$

Momentum

$$\rho(u_t + uu_x + vu_y + wu_z) = -p_{ex} + (\mu u_y)_y + (\mu u_z)_z \tag{2a}$$

$$\rho(v_t + uv_x + vv_y + wv_z) = -p_y + (\mu v_y)_y + (\mu v_z)_z \tag{2b}$$

$$\rho(w_t + uw_x + vw_y + ww_z) = -p_z + (\mu w_y)_y + (\mu w_z)_z \tag{2c}$$

where ρ denotes density, p pressure, μ coefficient of viscosity, t time, subscripts t,x,y,z denote partial differentiation with respect to the indicated variable, and subscript e denotes local conditions external to the viscous region (which may be a function of x and t).

It should be noted that the pressure gradient term in the axial momentum equation (2a) is denoted by the subscript e, which implies that if an axial pressure gradient does exist it must be known from an inviscid solution or coupled to it. Furthermore, it can be shown that, to the first-order approximation, p_y and p_z are of order $(u_\infty/\nu_\infty)^{-1/2}$, and, therefore, the pressure p can be considered constant in planes normal to the axial coordinate and equal in value to the external pressure at its corresponding station.

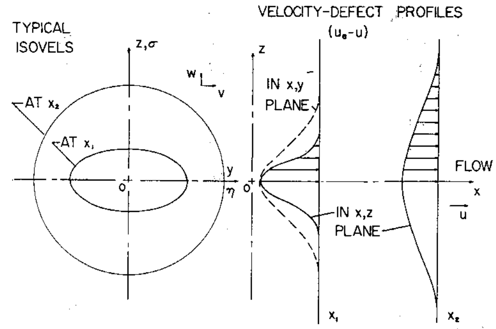

Fig. 1 Schematic diagram of axial flow field

For a compressible flow, the foregoing system must be supplemented by the stagnation enthalpy equation, conservation of the ith species equations, and the usual thermodynamic algebraic relations. For simplicity, when considering reacting mixtures of perfect gases (such as high temperature air), it will be assumed‡ that the binary diffusion coefficients of the several species are approximately the same, so that each diffusion velocity is given by a Fick's law form. Therefore, a single Lewis number appears, wherein the fluid properties are those of the mixture. Likewise, a single Prandtl number of the mixture is defined.

When considering compressibility, the system (1–2c) is completed by the following equations:

Energy

$$\rho[H_t + uH_x + vH_y + wH_z] - p_t = \left[\frac{\mu}{Pr} H_y\right]_y + \left[\frac{\mu}{Pr} H_z\right]_z + \left[\frac{\mu}{Pr}(Pr - 1)uu_y\right]_y + \left[\frac{\mu}{Pr}(Pr - 1)uu_z\right]_z + \left[\frac{\mu}{Pr}(Le - 1)\Sigma\, h_i\alpha_{i_y}\right]_y + \left[\frac{\mu}{Pr}(Le - 1)\Sigma\, h_i\alpha_{i_z}\right]_z \tag{3}$$

Conservation of Species

$$\rho[\alpha_{i_t} + u\alpha_{i_x} + v\alpha_{i_y} + w\alpha_{i_z}] = [(\mu/Pr)Le\ \alpha_{i_y}]_y + [(\mu/Pr)Le\ \alpha_{i_z}]_z + \rho W_i \tag{4}$$

Equation of State

$$p = \rho R_0 T \Sigma(\alpha_i/M_i) \tag{5}$$

Total Enthalpy

$$H = h + u^2/2 \tag{6}$$

Enthalpy-Temperature (classical statistical mechanics)

$$h = R_0 T\left[\sum_j \left(\frac{\alpha_j}{M_j}\right)\left(\Lambda_j + \frac{7}{2}\right) + \frac{5}{2}\sum_k \frac{\alpha_k}{M_k}\right] + \sum_l \alpha_l D_l \tag{7}$$

where i refers to each species, j refers to molecular (diatomic) species, k refers to atomic species, and l is associated with dissociation and ionization energies, and where H denotes stagnation enthalpy, h static enthalpy, Le Lewis number, Pr Prandtl number, α_i mass fraction of ith species, and T temperature. Variables not defined in the text are defined in the Nomenclature.

For reactions in chemical equilibrium, Eq. (4) is superfluous, since each W_i becomes indeterminate, being the prod-

‡ This approximation is discussed in Ref. 8, Appendix A.

uct of two terms, one of which (the reaction rate) approaches infinity while the other approaches zero. The terms that approach zero provide an algebraic set for the species concentration.

Equations (1–7) govern either laminar flow or the mean quantities of turbulent flow if, in the turbulent case, each transport parameter is interpreted as being the corresponding sum of the laminar transport coefficient and its turbulent eddy counterpart. However, turbulent considerations clearly are handicapped by a dearth of knowledge concerning the behavior of the eddy viscosity in three-dimensional flows. Here it is suggested that the eddy viscosity may be assumed in the form

$$\epsilon_v = K(2\bar{\delta}_m)\rho_r(u_e - u_0)$$

where $\bar{\delta}_m$ is taken to be the minimum transformed crosswise dimension. This gives the proper limiting behavior in the two-dimensional and axisymmetric limits.

B. Integrals of the Energy and Diffusion Equation in Terms of Velocity

For steady flow [i.e., $( )_t = 0$], direct consideration of the governing differential equations shows that, for the approximation $Le = Pr = 1.0$, the Crocco integral to the energy equation can be derived; i.e., for the pressure gradient case

$$H = H_e = \text{const} \tag{8a}$$

and for uniform pressure

$$H = A + Bu \tag{8b}$$

where A and B are constant satisfying the appropriate boundary conditions and subscript c denotes conditions at an initial station. The constants, for example, can be evaluated by prescribing that H equals H_{0c} and H_e when u equals u_{0c} and u_e, respectively. With these conditions Eq. (8b) can be expressed in the form

$$(H - H_e)/(H_{0c} - H_e) = (u - u_e)/(u_{0c} - u_e) \tag{8b$'$}$$

where subscript 0 denotes conditions along the x axis.

Likewise, when the chemical rates (ρW_i) are small compared to the transport processes and can be neglected, and moreover $Le/Pr = 1.0$, the following integrals to the species equation are derived, i.e., for the pressure gradient case:

$$\alpha_i = \alpha_{i_e} = \text{const} \tag{9a}$$

and for uniform pressure

$$(\alpha_i - \alpha_{ie})/(\alpha_{i0c} - \alpha_{i_e}) = (u - u_e)/(u_{0c} - u_e) \tag{9b}$$

Equation (9b) is comparable to (8b′) and satisfies the conditions $\alpha_i = \alpha_{i_{0c}}$ and α_{i_e} when $u = u_{0c}$ and u_e.

These integrals are valid for either laminar or turbulent flows, as long as the proper interpretation is given to the Lewis and Prandtl numbers.

C. Integral Flux Equations for the Axial Velocity

It is of interest now to present an infinite set of integral equations that involve only the axial velocity and the state variables. This is achieved by multiplying (2a) by u^m (where m is a non-negative integer), integrating over the y,z plane, and using the continuity equation. The following set thus is derived:

$$\frac{1}{u_e}[\theta_t + \theta(\ln\rho_e u_e^{m+1})_t + \delta^*(\ln u_e^{m+1})_t] + \theta_{E_x} + \theta_E(\ln\rho_e u_e^{m+2})_x + \delta_E^*(\ln u_e^{m+1}) = -\frac{(m+1)\mu_e}{\rho_e u_e}\int_0^\infty\int_0^\infty \bar{u}^m\{(\bar{\mu}\bar{u}_y)_y + (\bar{\mu}\bar{u}_z)_z\}dydz \tag{10}$$

where $\theta_E = \int_0^\infty\int_0^\infty \bar{\rho}\bar{u}(1 - \bar{u}^{m+1})dydz$, $\delta^* = \int_0^\infty\int_0^\infty (\bar{u}^m - \bar{\rho})dydz$, $\theta = \int_0^\infty\int_0^\infty \bar{\rho}(1 - \bar{u}^{m+1})dydz$, $\delta_E^* = \int_0^\infty\int_0^\infty (\bar{u}^m - \bar{\rho}\bar{u})dydz$, bars denote nondimensional quantities ($\bar{\rho} = \rho/\rho_e$, $\bar{\mu} = \mu/\mu_e$, $\bar{u} = u/u_e$), and it has been assumed that the following relation is valid:

$$\rho_e[u_{e_t} + u_e u_{e_x}] = -p_{e_x} \tag{10a}$$

Equation (10) is essential in the analysis of three-dimensional free-mixing problems. It may be used in a number of different ways, for example:

1) Part of the set can be used in conjunction with one or more auxiliary conditions in deriving solutions by the well-known integral method. An auxiliary condition can be, for example, satisfaction of the axial momentum equation along the x axis. The number of equations used must be equal to the number of parameters in the assumed profile, and, therefore, with the numerous equations available there exists a large degree of freedom in selecting the profile.

2) A major problem in three-dimensional free mixing is to account for compressibility, since there does not appear to be a density transformation that would reduce the differential equations to an incompressible form. However, it may be recalled that, for an axisymmetric flow, a transformation does exist which reduces the governing integral equations to a more suitable form and, in particular, for a steady isobaric flow this form is identical to that of an incompressible flow. In the three-dimensional case an analogous result is obtained. Clearly, this admits an approximate method by which known exact incompressible solutions can be extended to account for compressibility.

3) Finally, the set can be used to derive higher approximations to the solutions.

In this paper use is made of method 2. Second approximations are presented in Ref. 1, whereas solutions generated by the integral method are left for future work.

Equation (10) also admits an important result; namely, for a steady isobaric flow the condition $m = 0$ results in the following relation:

$$\int_0^\infty\int_0^\infty \overline{\rho u}(1 - \bar{u})\,dydz = \text{const} \tag{11}$$

which states that the momentum defect integral is invariant. Thus, the well-known result for two-dimensional and axisymmetric flows is extended to the more general three-dimensional case. In addition, Eq. (11) is valid for compressible laminar or turbulent flow and for wakes or jets.

D. Governing Equations in the Asymptotic Downstream Region

To achieve additional simplicity, flows are considered wherein decrements or increments with respect to the inviscid velocity u_e are sufficiently small to permit the linearization of the acceleration terms (excluding density) in the equations of motion. The linearization is in the familiar spirit of Oseen and others.

By defining the velocity defect $\tilde{u} = u_e - u$ and by assuming $\tilde{u} \ll u_e$, the governing boundary layer equations become the following:

Continuity

$$\rho_t + (\rho v)_y + (\rho w)_z = (\rho\tilde{u})_x - (\rho u_e)_x \tag{12}$$

Momentum

$$\rho[(\tilde{u} - u_e)_t + (u_e\tilde{u})_x - u_e u_{e_x}] = p_{e_x} + (\mu u_y)_y + (\mu u_z)_z \tag{13a}$$

$$\rho[v_t + u_e v_x] = -p_y + (\mu v_y)_y + (\mu v_z)_z \tag{13b}$$

$$\rho[w_t + u_e w_x] = -p_z + (\mu w_y)_y + (\mu w_z)_z \tag{13c}$$

It is of interest to note that Eq. (13a), which governs the axial velocity, is uncoupled from those governing the crossflow components.

E. Mathematically Exact Solutions for Incompressible Flow

For a constant-property flow, Eqs. (12) and (13) are linear, and, therefore, a solution can be derived by using transform methods. Hereafter the discussion is limited to steady flows.§

1. Axial flow including streamwise pressure gradient

Equation (13a) is recast as follows:

$$Q_s = Q_{\eta\eta} + Q_{\sigma\sigma} \tag{14}$$

where

$$Q = (\tilde{u}u_e/u_\infty^2) - [(p/\rho u_\infty^2) + \tfrac{1}{2}(u_e^2/u_\infty^2)] \tag{14a}$$

$$\frac{x - x_c}{L} = \int_0^s \left[\frac{\mu}{\mu + \epsilon_v}\right]^j \frac{u_\infty}{u_e}\, ds \tag{14b}$$

$$\mu = (y/L)(Lu_\infty/\nu_\infty)^{1/2} \qquad \sigma = (z/L)(Lu_\infty/\nu_\infty)^{1/2} \tag{14c}$$

where $j = 0$ for laminar flows, $j = 1$ for turbulent flows, L defines a constant characteristic length, and subscript ∞ denotes freestream (undisturbed) flight conditions. In obtaining (14), it is recalled that the pressure p and the external velocity u_e are only functions of x. The bracketed term in Eq. (14a) signifies the local stagnation pressure of the outer flow.

Equation (14) shows that in terms of the transformed variables the analyses for laminar and turbulent flows are identical.

The appropriate boundary conditions are

$$\text{at } s = 0 \qquad \tilde{u} = \tilde{u}_c(\eta,\sigma) \tag{15a}$$

$$\text{as } s \to \infty \qquad \tilde{u} \to 0 \tag{15b}$$

and

$$\left.\begin{array}{ll} \text{as } \eta \to \infty & \text{or } \eta \to -\infty \\ \sigma \to \infty & \sigma \to -\infty \end{array}\right\} \quad \tilde{u} \to 0 \tag{15c}$$

The solutions for $Q(s,\eta,\sigma)$ and the crossflow velocities v and w, in terms of Fourier transforms, have the same form as those given in Ref. 1.

2. A crossflow solution

Wavelike or jetlike free mixing for a steady, incompressible, pressure gradient flow with nonaxisymmetric small‖ swirl has been examined in Ref. 6, and the essential features of the analysis are indicated briefly here.

There are several interesting results worth noting. It is shown[6] that for small swirl the pressure difference in the cross planes is of higher order, and therefore the pressure gradient term in the momentum balance for the circumferential velocity does not appear. In addition, the equation governing the axial velocity is uncoupled from the one governing the swirl, the latter being affected only through u_e.

The boundary layer equation (in cylindrical coordinates) for the circumferential velocity component is shown to reduce to the following:

$$V_s = V_{nn} + (1/n)V_n - (V/n^2) + (1/n^2)V_{\theta\theta} \tag{16}$$

where $n = (r/L)(u_\infty L/z_\infty)^{1/2}$, r and θ denote the radial and circumferential coordinates, respectively, V is the circumferential velocity component, and the other variables have been defined previously.

Equation (16) is linear in V, and therefore solutions may be facilitated by the use of transforms. Details can be found in Ref. 6.

F. Approximate Compressible Solutions for the Axial Velocity Distribution

In the previous section, it was shown that for a constant property flow the governing equations can be linearized, and, therefore, mathematically exact solutions can be derived by transform methods. On the other hand, when compressibility must be accounted for, the equations are nonlinear and analogous solutions cannot be achieved. However, useful and well-defined approximate techniques are available. For example, in Ref. 2 Steiger and Bloom use an iteration method in which the nonlinear terms (that is, terms due to the compressibility) are assumed to be known functions of the coordinates, whereby the governing equation for the axial velocity becomes linear but nonhomogeneous and, therefore, amenable to solution by transform methods. The iteration then involves an evaluation of several integrals. However, this approach has several shortcomings, namely, the integrals involved in the iteration are extremely cumbersome and only one iteration is feasible, the solution can be given in closed form only along the axis, the error incurred in the iteration cannot be determined, and there is a range of parameters where the iteration diverges. It can be concluded that this technique yields reasonably good results only when compressibility slightly alters the flow.

Here an approach is presented which is believed to give more reliable results and accuracy. The method involves the extrapolation of the incompressible solutions presented in the previous section to compressible ones by means of suitable coordinate transformations. The transformations are based on the integral method and, as in the well-known axisymmetric case,[8] they may be expected to involve both the density and the viscosity.

For brevity the discussion is restricted to a steady isobaric flow. Consider the axial momentum flux integral (11) and the nonlinear momentum equation evaluated at the x axis, namely,

$$\int_0^\infty \int_0^\infty \frac{\rho}{\rho_e}\frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dydz = \left[\int_0^\infty \int_0^\infty \frac{\rho}{\rho_e}\frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dydz\right]_c \tag{17a}$$

and

$$\rho_0 u_0 u_{0_x} = \mu_0[u_{yy_0} + u_{zz_0}] \tag{17b}$$

It is desired to generate transformations that will reduce (17) to an incompressible form. For this purpose, let

$$y = \beta \qquad z = \left(\frac{\nu_e L}{u_e}\right)^{1/2} \int_0^\xi \left[\frac{\rho_e}{\rho(\beta,\xi)}\right]^{1/2} d\xi \tag{18}$$

where $\rho(\beta,\xi)$ implies that the density is a function of β and ξ. The Jacobian of the transformation is

$$J(y,z/\beta,\xi) = (\nu_e L/u_e)^{1/2}(\rho_e/\rho)^{1/2}$$

Therefore, Eq. (17a) is reduced to

$$\left(\frac{\nu_e L}{u_e}\right)^{1/2} \int_0^\infty \int_0^\infty (\bar{\rho})^{1/2}\bar{u}(1 - \bar{u})d\beta d\xi = \text{const} \tag{19}$$

Likewise, introduce

$$\beta = \left(\frac{\nu_e L}{u_e}\right)^{1/2} \int_0^{\eta_1} \left[\frac{\rho_e}{\rho(\eta_1,\sigma_1)}\right]^{1/2} d\eta_1 \qquad \xi = \sigma_1 \tag{20}$$

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§ Unsteady flows are omitted here to limit the scope of the paper rather than because of any difficulties associated with their solution. Such flows are under investigation at the present time.

‖ The order of magnitude associated with small swirl is discussed in Refs. 6 and 9.

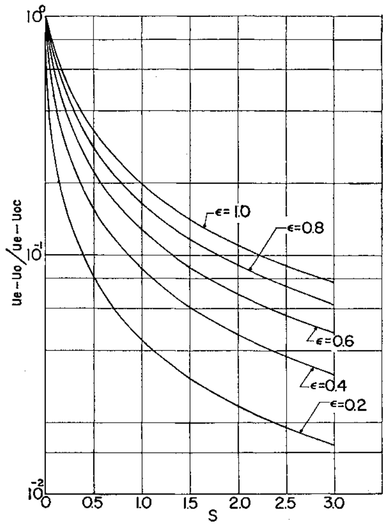

Fig. 2 Distribution of the velocity defect along the s axis

and, since $J(\beta,\xi/\eta_1,\sigma_1) = (\nu_e L^{1/2}/u_e)\ (\rho_e/\rho)^{1/2}$, Eq. (19) is transformed to

$$\frac{\nu_e L}{u_e}\int_0^\infty\int_0^\infty \bar{u}(1-\bar{u})d\sigma_1 d\eta_1 = \text{const} \tag{21}$$

noting that Eq. (21) is independent of the density.

In addition, by means of (18) and (20) and the following relation:

$$x - x_c = L\int_0^{s_1}\left[\frac{\mu_0}{\epsilon_v + \mu_0}\right]^j \frac{\mu_e}{\mu_0}ds_1 \tag{22}$$

Equation (17b) is reduced to

$$\bar{u}_0\bar{u}_{0_{s_1}} = \bar{u}_{\eta_1\eta_{1_0}} + \bar{u}_{\sigma_1\sigma_{1_0}}$$

Equations (18, 20, and 22) are considered here to furnish the appropriate coordinate transformations that apply rigorously in integral method solutions but that may be used also in a semi-empirical manner to extrapolate constant-property results to the compressible regime.

Extreme care should be exercised in transforming the solution to the physical plane. This is because a direct relation between the transverse physical (y,z) and transformed (η,σ) coordinates does not exist. This is seen by noting that in (20) the density is a function of the coordinates η and σ, whereas in (18) it is a function of β and ξ. Indeed, in order to evaluate the appropriate values of y and z, one first must use (20) and then apply (18) to complete the transformation. On the other hand, the streamwise physical coordinate (x) is directly related to (s) by (22).

For a constant-property flow, $\eta_1 = \eta$, $\sigma_1 = \sigma$, and $s_1 = s$. Therefore, the subscript 1 will be omitted hereafter.

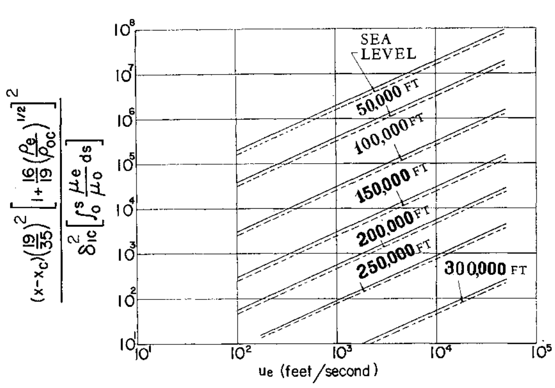

Fig. 3 Physical length parameter for laminar flow; —— $u_{0c} = 0.4u_e$, - - - - $u_{0c} = 0.7u_e$

III. Fluid Mechanics: Solutions for Special Configurations

A. Elliptic Wakelike or Jetlike Flows; Axial Velocity

The flow is assumed to be steady and isobaric. The incompressible solutions have been presented in Ref. 1 and are achieved in the following manner. At an initial station, let

$$(u_e - u_c)/(u_e - u_{0_c}) = e^{-[\eta^2+(\sigma/\epsilon)^2]} \tag{23}$$

where ϵ is a pure constant that lies in the range $0 < \epsilon \leq 1$ and defines the eccentricity of the isovels (lines of constant velocity). Subscript e denotes conditions at the edge of the viscous layer which, in the present case, are constant and, therefore, can be identified with the freestream conditions.

The distribution of the axial velocity defect for $s \geq 0$ satisfying the boundary conditions (15) can be shown to be

$$(u_e - u)/(u_e - u_{0_c}) = \varphi_0 e^{-k^2[\eta^2+(\sigma/E)^2]} \tag{24}$$

where $k^2 = [1 + 4s]^{-1}$, $E^2 = k^2[\epsilon^2 + 4s]$, and φ_0 is the distribution of velocity along the axis and is given by

$$\varphi_0 = \frac{\epsilon}{(1+4s)^{1/2}(\epsilon^2+4s)^{1/2}} \tag{25}$$

Figure 2 shows the variation of the velocity defect (25) along the axis for various values of ϵ.

The extrapolation to the compressible regime is achieved by using Eqs. (18, 20, and 22).

The characteristic length (L) is, in general, defined by the initial conditions, that is, by using a familiar definition of a viscous layer thickness, namely, in the plane $z = 0$, $y = \delta_1$, when $u = u_1 = 0.99u_e$. With Eqs. (18, 20, and 23) it is seen readily that

$$L = \frac{\delta_{1c}^2 u_e}{\nu_e}\left[\int_0^{\eta_\delta}\left\{\frac{\rho_e}{\rho(\eta,\sigma=0)}\right\}^{1/2} d\eta\right]_c^2 \tag{26}$$

where

$$\eta_{\delta_c}^2 = \ln\{100[(u_e - u_{0_c})/u_e]\} \tag{26a}$$

IV. Discussion of the Solution

Some results of the solution of the compressible, steady, uniform pressure elliptic wake are presented first for laminar flows and then for fully turbulent flows. The solution is given by Eqs. (5–7, 8b′, 9b, 18, 20, 22, and 24–26).

A. Laminar Flow

In the laminar case $j = 0$, and Eq. (22) yields, for the relation between the physical and transformed streamwise coordinate, the following:

$$x - x_c = \frac{u_e\delta_{1c}^2}{\nu_e}\frac{\int_0^s \frac{\mu_e}{\mu_0}ds}{\left[\int_0^{\eta_\delta}\left\{\frac{\rho_e}{\rho(\eta,\sigma=0)}\right\}^{1/2}d\eta\right]_c^2} \tag{27}$$

Equation (27) is consistent with the qualitative reasoning, which indicates that the diffusion times of order δ^2/ν and the corresponding diffusion lengths of order $\delta^2 u_e/\nu$ will be shortened by the effects of compressibility, which increases the mean value of ν.

It is of interest to estimate a parametric set of curves to assist in the evaluation of laminar wake lengths. To achieve simplicity and generality in executing the required transformation, an approximate profile for the density is assumed directly in lieu of the more precise but cumbersome exact expression,# namely

Several numerical exact calculations indicated that (28) is a reasonably good approximation.

$$\left[\frac{\rho_e}{\rho(\eta,\sigma=0)}\right]^{1/2} = \left(\frac{\rho_e}{\rho_0}\right)^{1/2} + \left[1 - \left(\frac{\rho_e}{\rho_0}\right)^{1/2}\right] \times [3\bar{\eta}^2 - 3\bar{\eta}^4 + \bar{\eta}^6] \quad (28)$$

where $\bar{\eta} = \eta/\eta_{\delta_1}$, and η_{δ_1} is the value of η where $y = \delta$ in the plane $z = 0$.

With Eq. (28), it follows that

$$\left[\int_0^{\eta_\delta} \left(\frac{\rho_e}{\rho}\right)^{1/2} d\eta\right]_c^2 = \left(\frac{19}{35}\right)^2\left[1 + \frac{16}{19}\left(\frac{\rho_e}{\rho_{0_c}}\right)^{1/2}\right]^2 \times \left[\ln\left(100\,\frac{u_e - u_{0_c}}{u_e}\right)\right] \quad (29)$$

Therefore (27) reduces to

$$x = x_c + \frac{u_e\delta_{1c}^2}{\nu_e} \times \frac{\int_0^s \frac{\mu_e}{\mu_0}\,ds}{\left(\frac{19}{35}\right)^2\left[1 + \frac{16}{19}(\rho_e/\rho_{0_c})^{1/2}\right]^2\left[\ln\left(100\,\frac{u_e - u_{0_c}}{u_e}\right)\right]} \quad (30)$$

Figure 3 presents the variation of the physical length parameter,

$$\frac{(x - x_c)\left(\frac{19}{35}\right)^2\left[1 + \frac{16}{19}\left(\frac{\rho_e}{\rho_{0_c}}\right)^{1/2}\right]^2}{\delta_{1c}^2 \int_0^s \frac{\mu_e}{\mu_0}\,ds}$$

for various flight conditions and initial velocities at the axis.

It is observed that, when $4s \gg \epsilon^2$, Eq. (24) reduces to

$$(u_e - u)/(u_e - u_{0_c}) = (\epsilon/2s)e^{-(1/4s)(\eta^2+\sigma^2)} \quad (31)$$

which has the well-known axisymmetric character. Although (31) is given in terms of (s,η,σ), it is expected that the results also should be valid in the physical plane, since the distortion of the isovels due to variations in density is small when $s \gg 1.0$. Therefore, the result that, in an incompressible flow, a wake with any degree of initial eccentricity ultimately degenerates to an axisymmetric configuration and mode of decay also is valid for a compressible flow.

The distance required for the ellipticity (e) to decay to a specified value may be obtained by using (28) and a similar equation for $\rho(\eta = 0,\sigma)$, namely

$$\left[\frac{\rho_e}{\rho(\eta=0,\sigma)}\right]^{1/2} = \left(\frac{\rho_e}{\rho_0}\right)^{1/2} + \left[1 - \left(\frac{\rho_e}{\rho_0}\right)^{1/2}\right] \times [3\bar{\sigma}^2 - 3\bar{\sigma}^4 + \bar{\sigma}^6] \quad (32)$$

where $\bar{\sigma} = \sigma/\sigma_{\delta_2}$, and σ_{δ_2} is the value of σ when $z = \delta$ in the plane $y = 0$.

With

$$\eta_{\delta 1} = (1 + 4s)\ln\{100[(u_e - u_0)/u_e]\} \quad (33a)$$

and

$$\sigma_{\delta 1} = (\epsilon^2 + 4s)\ln\{100[(u_e - u_0)/u_e]\} \quad (33b)$$

it follows that

$$\delta_1 = \delta_{1c}(1 + 4s)^{1/2}\left[\frac{1 + \frac{16}{19}(\rho_e/\rho_0)^{1/2}}{1 + \frac{16}{19}(\rho_e/\rho_{0_c})^{1/2}}\right] \times \left[\frac{[\ln\{100[(u_e - u_0)/u_e]\}}{[\ln\{100[(u_e - u_{0_c})/u_e]\}}\right]^{1/2} \quad (34)$$

and

$$\delta_2 = \delta_1[(\epsilon^2 + 4s)/(1 + 4s)]^{1/2} \quad (35)$$

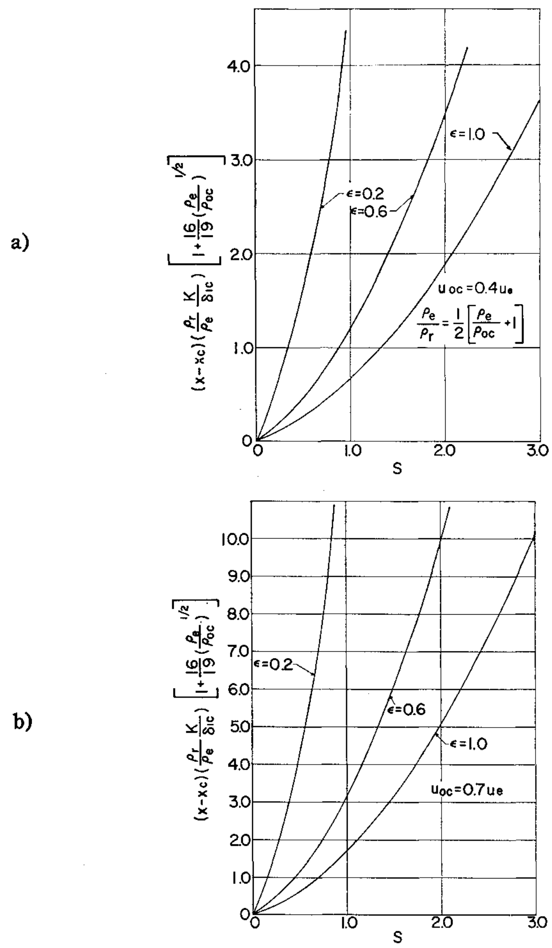

Fig. 4 Physical length parameter for turbulent flow

The ratio of the thicknesses in terms of s given by (35) is independent of explicit density effects. The influence of compressibility appears in the distortion of $s(x)$ given by Eq. (27).

Since the eccentricity (e) is defined by

$$e = [1 - (\delta_2/\delta_1)^2]^{1/2} \quad (36)$$

it follows that

$$s = (e_c^2 - e^2)/4e^2 \qquad e_c = (1 - \epsilon^2)^{1/2} \quad (37)$$

Equation (37) gives the distance (s) required for a wake with an initial eccentricity e_c to decay to a wake with an eccentricity e.

B. Turbulent Flow

For fully turbulent flows, it is assumed that the eddy viscosity ϵ_v is at most a function of the streamwise coordinate. To obtain an estimate of the change in the streamwise scale due to turbulence, a relation of the following form is proposed for the eddy coefficient of viscosity:

$$\epsilon_v = K\rho_r(2\bar{\delta}_m)(u_e - u_0) \quad (38)$$

where $\bar{\delta}_m$ is the minimum transformed viscous layer thickness, namely

$$\bar{\delta}_m = (L\nu_e/u_e)^{1/2}\sigma_{\delta_1} \quad (39)$$

It is noted that $\bar{\delta}_m = \delta_e$ when $\rho = $ const.

This eddy viscosity model properly represents the limiting cases of two-dimensional and axisymmetric flows. (It is considered to be more reasonable in this regard than the model posed in Ref. 1.)

With Eqs. (26, 29, 38) and the assumption that the laminar viscosity is negligible compared to its turbulent counterpart, it follows that

$$\frac{x - x_c}{\delta_{1c}} = \frac{\int_0^s \frac{\rho_e}{\rho_r}\left(\frac{u_e}{u_e - u_0}\right)\frac{ds}{\sigma_{\delta_1}}}{\frac{38}{35}K\left[1 + \frac{16}{19}\left(\frac{\rho_e}{\rho_{0c}}\right)^{1/2}\right]\left[\ln\left(100\,\frac{u_e - u_{0c}}{u_e}\right)\right]^{1/2}} \qquad (40)$$

where $(u_e - u_0)/u_e = [(u_e - u_{0c})/(u_e)]\varphi_0$, and $\sigma_{\delta 1}$ is given by Eq. (33b).

Relation (40) is independent of Reynolds number. In the absence of experimental evidence on the appropriate evaluation of ρ_r, it is assumed that

$$1/\rho_r = \tfrac{1}{2}[(1/\rho_{0c}) + (1/\rho_e)] \qquad (41)$$

On this basis (40) indicates that compressibility increases the streamwise scale. Indeed, one can write

$$\frac{(x - x_c)_{\text{compressible}}}{(x - x_c)_{\text{incompressible}}} = \frac{35}{38}\,\frac{[1 + \rho_e/\rho_{0c}]}{[1 + \frac{16}{19}(\rho_e/\rho_{0c})^{1/2}]} \qquad (42)$$

The right-hand side of (42) is greater than or equal to unity for $\rho_e/\rho_{0c} \geq 1.0$. For example, for $\rho_e/\rho_{0c} = 9.0$, the ratio is 2.61.

Figure 4 presents the variation of the physical length parameter

$$[(x - x_c)/\delta_{1c}][K(\rho_r/\rho_e)][1 + \tfrac{16}{19}(\rho_e/\rho_{0c})^{1/2}]$$

for various values of u_{0c} and ϵ.

Finally, it is noted that the conclusions concerning Eqs. (31–37) for laminar flow can be carried over to the turbulent case. However, Eq. (40) must be used to determine the physical length $(x - x_c)$.

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Calendar of Forthcoming AIAA Meetings

Date	Meeting	Location	Abstract Deadline
April 22–24	2nd AIAA-NASA Manned Space Flight Meeting	Dallas, Texas	Past
April 23–25	AIAA-ASME Hypersonic Ramjet Conference[1]	White Oak, Md.	Past
May 2	AIAA-ASMA Manned Space Laboratory Conference[2]	Los Angeles, Calif.	Past
May 6–8	AIAA-ASME-SAE Aerospace Reliability and Maintainability Conference	Washington, D. C.	Past
May 20–22	AIAA-IEEE-ISA National Telemetering Conference	Albuquerque, New Mex.	Past
June 12–14	Heat Transfer and Fluid Mechanics Institute[3]	Pasadena, Calif.	Past
June 17–20	AIAA Summer Meeting[2]	Los Angeles, Calif.	Past
July 10–12	AIAA-AMS Meteorological Support for Aerospace Testing and Operation	Fort Collins, Colo.	Past
July 23–26	AIAA Torpedo Propulsion Conference[4]	Newport, R. I.	Past
Aug. 12–14	AIAA Guidance and Control Conference[5]	Cambridge, Mass.	Past
Aug. 19–21	AIAA Astrodynamics Conference[2]	New Haven, Conn.	Past
Aug. 26–28	AIAA Simulation for Aerospace Flight Conference[2]	Columbus, Ohio	Past
Aug. 26–28	AIAA Conference on Physics of Entry into Planetary Atmospheres[5]	Cambridge, Mass.	Past
Sept. 22–27	XIVth International Astronautical Congress[6]	Paris, France	Past
Sept. 23–27	1st International Telemetering Conference	London, England	Past
Sept. 30–Oct. 1	AIAA Engineering Problems of Manned Interplanetary Exploration Meeting[2]	Palo Alto, Calif.	Past
Nov. 4–6	AIAA Vehicle Design and Propulsion Meeting[2]	Dayton, Ohio	Past
Dec. 4–6	AIAA-Air Force Testing of Manned Flight Systems	Edwards Air Force Base, Calif.	Past

[1] Call for papers in December issue. Program available from AIAA New York office.
[2] Flyer with additional program details available from AIAA New York office.
[3] Call for papers in October issue.
[4] Call for papers in March issue. Flyer with additional program details available from AIAA New York office.
[5] Call for papers in February issue. Flyer with additional program details available from AIAA New York office.
[6] Call for papers in January issue and in this issue.

Except when program chairman is known, abstracts should be mailed to Robert R. Dexter, Director of Technical Services, American Institute of Aeronautics and Astronautics, 500 Fifth Avenue, New York 36, N. Y.

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